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An International Journal
computers & mathematics
with applications

ELSEVIER Computers and Mathematics with Applications 48 (2004) 429–435

www.elsevier.com/locate/camwa

A Numerical Verification of Solutions of Free Boundary Problems

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(Received December 2003; accepted January 2004)

Abstract—We consider numerical verification techniques which enable us to verify the existence of solutions for the free boundary problems governed by elliptic variational inequalities of the first kind.

Keywords—Numerical verification, Free boundary problems, Fixed-point theorem, Error estimates, Variational inequalities.

1. INTRODUCTION

In this article, we describe the verification method for solutions of the obstacle problem, which is known as a free boundary problem characterizing the contacted zone with the obstacle of an elastic membrane. To solve an obstacle problem with automatic verification of correctness of the result, a nonlinear system of equations as in [1] could be used. However, taking advantage of the special structure of the obstacle problem finally leads to a system of linear equations. Using the system of linear equations, we propose another approach to verify the existence of solutions for an obstacle problem with convex set $K = \{v \in H_0^1(\Omega) : v \geq \psi\}$ and ψ is the height of the obstacle.

Let Ω be a bounded convex domain in $\mathbf{R}^n$, $1 \leq n \leq 2$, with piecewise smooth boundary Γ. We set $H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_\Gamma = 0\}$ and

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx, \qquad \text{where } \nabla u \cdot \nabla v = \frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1} + \frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2}.$$

For $u, v \in H_0^1(\Omega)$, we have $a(u,v) = (\nabla u, \nabla v)$, where $(\cdot,\cdot)$ denotes the L^2-inner product on Ω. Denote $\|\cdot\|_{L^2(\Omega)}$ as the usual L^2 norm on Ω. Next, we define $K = \{v \in H_0^1(\Omega) : v \geq \psi$ a.e. on $\Omega\}$, here, ψ is a given function in $H^2(\Omega)$, such that $\psi \leq 0$ on Γ. We note that, for any $g \in L^2$, the problem

$$a(u, v-u) \geq (g, v-u), \qquad \forall\, v \in K, \quad u \in K, \tag{1.1}$$

This work was supported by Korea Research Foundation Grant KRF-2002-015-CP0046.

0898-1221/04/$ - see front matter Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX
doi:10.1016/j.camwa.2004.01.008

has a unique solution $u \in H_0^1(\Omega) \cap H^2(\Omega)$, and the estimate

$$|u|_{H^2(\Omega)} \leq \max\left\{\|g\|_{L^2(\Omega)}, \sigma\right\} \tag{1.2}$$

holds. Here σ is the solution of the equation

$$\sigma = \|\max(A\psi, 0)\|_{L^2(\Omega)} \frac{\left(\sigma + \|g\|_{L^2(\Omega)}\right)}{\left(\sigma - \|g\|_{L^2(\Omega)}\right)}$$

and $|u|_{H^2}$ implies the seminorm of u on $H^2(\Omega)$. We adopt $(\nabla\varphi, \nabla v)$ as the inner product on $H_0^1(\Omega)$. Hence, the associated norm is defined by $\|\varphi\|_{H_0^1(\Omega)} = \|\nabla\varphi\|_{L^2(\Omega)}$.

We consider the following obstacle problem:

$$\text{find } u \in K, \text{ such that } a(u, v-u) \geq (f(u), v-u), \qquad \forall\, v \in K. \tag{1.3}$$

In what follows, the map f as in (1.3) is assumed to be continuous from the Sobolev space $H_0^1(\Omega)$ into $L^2(\Omega)$, such that it has a bounded image in $L^2(\Omega)$ on a bounded set in $H_0^1(\Omega)$. To verify the existence of a solution of (1.3) in a computer, we use the fixed point formulation. As in the preceding paper [1], defining the projection $P_K : H_0^1(\Omega) \to K$, problem (1.3) is equivalent to that of finding $u \in H_0^1(\Omega)$, such that

$$u = P_K F(u). \tag{1.4}$$

Here, $F(u) \in H_0^1(\Omega)$, such that $-\Delta F(u) = f(u)$ in Ω, $F(u) = 0$ on Γ. Then the map $P_K F$ is a compact operator by the above assumptions on f.

Schauder's fixed point theorem yields the existence of a solution u of problem (1.4) in some suitable set $U \subset H_0^1(\Omega)$, provided that

$$P_K F(U) \subset U. \tag{1.5}$$

In order to compute an explicit inclusion, we must therefore construct U. Now we describe how to construct U explicitly. First, we determine a set $W \equiv P_K F(U)$ for a bounded, convex, and closed subset $U \subset H_0^1(\Omega)$ as

$$W = \left\{w \in H_0^1(\Omega) : w = P_K F(u),\ \forall\, u \in U\right\}. \tag{1.6}$$

From Schauder's fixed point theorem, if $W \subset U$ holds, then there exists a solution of (1.3) in the set U. A procedure to verify $W \subset U$ using a computer is similar to that in [1] as below. Now, let V_h be a finite dimensional subspace of $H_0^1(\Omega)$ dependent on h $(0 < h < 1)$ and let K_h be a nonempty closed convex subset of V_h. Letting $\mathcal{N}$ denote the set of nodes associated with the space V_h, we then define K_h, an approximation of K, by $K_h = \{v_h \in V_h : v_h(p) \geq \psi(p),\ \forall\, p \in \mathcal{N}\}$. Note that usually $K_h \not\subset K$, namely, we use an outer approximation.

We now consider the approximate problem of (1.1)

$$\text{find } u_h \in K_h, \text{ such that } a(u_h, v_h - u_h) \geq (g, v_h - u_h), \qquad \forall\, v_h \in K_h. \tag{1.7}$$

Using (1.1),(1.7), and error estimates, we make the following assumption.

ASSUMPTION. *For each $u \in H_0^1(\Omega)$, there exists a $C(g,\psi)$, such that*

$$\|u - u_h\|_{L^2(\Omega)} \leq C(g,\psi)h. \tag{1.8}$$

In order to verify the solutions numerically, it is necessary to determine the constant $C(g,\psi)$. This $C(g,\psi)$ will be discussed later.

Now, we introduce two concepts, rounding and rounding error. For any $u \in H_0^1(\Omega)$, we define the rounding $R(P_K F(u)) \in K_h$ as the solution of the following problem:

$$a(R(P_K F(u)), v_h - R(P_K F(u))) \geq (f(u), v_h - R(P_K F(u))), \qquad \forall\, v_h \in K_h.$$

We define the rounding $R(W) \subset K_h$ for the set W defined by (1.6) as

$$R(W) = \{w_h \in K_h : w_h = R(P_K F(u)),\ u \in U\}.$$

Also, we define the rounding error $RE(W) \subset H_0^1(\Omega)$ as

$$RE(W) = \left\{ w \in H_0^1(\Omega) : \|w\|_{H_0^1(\Omega)} \le \sup_{u \in U} C(f(u), \psi) h \right\}. \tag{1.9}$$

From the definition, we have

$$W \subset R(W) + RE(W), \tag{1.10}$$

which is the basic principle of our verification method. Then it is sufficient to find U satisfying $R(W) + RE(W) \subset U$.

In order to construct the set U satisfying the verification condition (1.10) in a computer, we use an iterative procedure, that is, the sequential iteration. We propose a computer algorithm to obtain the set U which satisfies condition (1.10).

(1) First, we obtain an approximate solution $w_h^{(0)} \in K_h$ to (1.7) by some appropriate method. Set $U^{(0)} = \{w_h^{(0)}\}$ and $\alpha_0 = 0$.

(2) Next we will define $R(W^{(i)})$ and $RE(W^{(i)})$ for $i \ge 0$, where $W^{(i)}$ is the set defined as follows:

$$W^{(i)} = \left\{ w^{(i)} \in K : w^{(i)} = P_K F\left(u^{(i)}\right),\ u^{(i)} \in U^{(i)} \right\}.$$

In order to enclose $W^{(i)}$, let us define $R(W^{(i)})$ as follows. $R(W^{(i)})$ is defined by the subset of K_h which consists of all elements $w_h^{(i)} \in K_h$, such that

$$a\left(w_h^{(i)}, v_h - w_h^{(i)}\right) \ge \left(f\left(u^{(i)}\right), v_h - w_h^{(i)}\right), \qquad \forall v_h \in K_h,$$

holds for some $u^{(i)} \in U^{(i)}$. Note that $R(W^{(i)})$ can be enclosed by $R(W^{(i)}) \subset \sum_{j=1}^{M} A_j \phi_j$, where $A_j = [\underline{A_j},\ \overline{A_j}]$ are intervals, $\{\phi_j\}_{j=1}^{M}$ is a basis of V_h, and $M = \dim V_h$. Secondly, $RE(V^{(i)})$ is defined by

$$RE\left(W^{(i)}\right) = \left\{ w \in H_0^1(\Omega) : \|w\|_{H_0^1(\Omega)} \le \sup_{u^{(i)} \in U^{(i)}} C\left(f\left(u^{(i)}\right), \psi\right) h \right\}.$$

Hence, $W^{(i)} \subset R(W^{(i)}) + RE(W^{(i)})$ holds.

(3) Check the verification condition

$$R\left(W^{(i)}\right) + RE\left(W^{(i)}\right) \subset U^{(i)}. \tag{1.11}$$

If the condition is satisfied, then $U^{(i)}$ is the desired set, and a solution to (1.3) exists in $W^{(i)}$, and hence in $U^{(i)}$.

(4) If the condition is not satisfied, we continue the simple iteration by using δ-inflation; i.e., let δ be a certain positive constant given beforehand, and take

$$\begin{aligned}
\alpha_{i+1} &= \sup_{u^{(i)} \in U^{(i)}} C\left(f\left(u^{(i)}\right), \psi\right) h + \delta, \\
[\alpha_{i+1}] &= \left\{ v \in H_0^1(\Omega) : \|v\|_{H_0^1(\Omega)} \le \alpha_{i+1} \right\}, \\
U_h^{(i+1)} &= \sum_{j=1}^{n} \left[\underline{A_j} - \delta, \overline{A_j} + \delta\right] \phi_j, \\
U^{(i+1)} &= U_h^{(i+1)} + [\alpha_{i+1}],
\end{aligned}$$

and then go back to the second step. If condition (1.11) is satisfied, in our inclusion method of solutions for (1.3), the solution u is enclosed in the set $U^{(i)}$, which we call 'a candidate set' of the form $U^{(i)} = U_h^{(i)} + [\alpha_i]$.

2. COMPUTING PROCEDURES

We propose a computer algorithm to obtain a set $U^{(i)}$ which satisfies the verification condition (1.11).

Since the bilinear form $a(\cdot,\cdot)$ is symmetric, (1.7) is actually equivalent to the quadratic programming problem

$$\min_{w_h \in K_h} \left[\frac{1}{2} a(w_h, w_h) - (g, w_h) \right]. \tag{2.1}$$

Let $z = (z_j) \in \mathbf{R}^M$ be the coefficient vector for $\{\phi_j\}$ corresponding to the function w_h in (2.1), and define $\hat{\psi} := (\psi_j) \in \mathbf{R}^M$, where $\psi_j = \psi(x_j)$, $j = 1, 2, \dots, M$. As parameters to describe a function $w_h \in V_h$ we choose the values $w_h(p_i)$ of w_h at the nodes p_i, $i = 1, \dots, M$, of $\mathcal{N}$ but exclude the nodes on the boundary since $w_h = 0$ on Γ. The corresponding basis functions $w_j \in V_h$, $j = 1, \dots, M$, are then defined by

$$\phi_j(x_i) = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

A function $w_h \in V_h$ now has the representation

$$w_h(t) = \sum_{j=1}^{M} z_j \phi_j(t), \quad z_j = w_h(p_j), \qquad \text{for } t \in \Omega \cup \Gamma.$$

Then we can represent the above quadratic programming problem (2.1) in the following form:

$$\min_{z \geq \hat{\psi}} \left[\frac{1}{2} z^\top D z - P^\top z \right]. \tag{2.2}$$

Here, $D = (\nabla\phi_i, \nabla\phi_j)$ and $1 \leq i, j \leq M$, and z is the coefficient vector for $\{\phi_j\}$ corresponding to the function w_h in (2.1). Furthermore, $P \equiv ((g, \phi_j))$ is an M-dimensional vector.

By the Kuhn-Tucker theorem [2], a vector $z - \hat{\psi} = (z_j) - (\psi_j) \in \mathbf{R}^M$ with $z - \hat{\psi} \geq 0$ is an optimal solution to (2.2) if and only if there exists $w = (w_j) \in \mathbf{R}^M$, such that

$$\begin{aligned} w - D(z - \hat{\psi}) &= D\hat{\psi} - (g, \phi_j), \qquad && 1 \leq j \leq M, \\ w(z - \hat{\psi}) &= 0, && w \geq 0, \quad z - \hat{\psi} \geq 0. \end{aligned} \tag{2.3}$$

Let $(\tilde{w}, \tilde{z} - \hat{\psi})$ be an approximate solution of (2.3). Let I, respectively, J be the set of indices i, respectively, j for which $\tilde{w}_i$, respectively, $\tilde{z}_j - \psi_j$ is approximately zero. Then delete in (2.3) every variable w_j, $z_j - \psi_j$ for which the corresponding component of $\tilde{w}$, $\tilde{z} - \hat{\psi}$ is approximately zero. Then M equations

$$w^* - D\left(z^* - \hat{\psi}\right) = D\hat{\psi} - P \tag{2.4}$$

remain, where w^*, z^* have on the whole M fewer components than w, z. Note that system (2.4) is linear.

Now, in order to evaluate the rounding $R(W)$ in Section 1, for a given set $U = \sum_{j=1}^{M} A_j \phi_j + [\alpha]$ and $g = f(U)$ in (2.3), we consider the linear system

$$w^* - D\left(z^* - \hat{\psi}\right) = D\hat{\psi} - (f(U), \phi_j), \qquad 1 \leq j \leq M. \tag{2.5}$$

Equation (2.5) is in fact a linear system of equations whose right-hand side consists of intervals.

Following [3], we have the following.

THEOREM 1. *Solve the nonlinear system (2.5) and let W_i, $1 \le i \le M$, $i \notin I$ and $Z_j - \psi_j$, $1 \le j \le M$, $j \notin J$ be the computed inclusions for the solutions. Define $W_i := 0$ for $i \in I$ and $Z_j - \psi_j := 0$ for $j \in J$ and let $W := (W_1, W_2, \ldots, W_n) \in \mathbf{IR}^M$ (real interval vectors with M components) and $Z - \hat{\psi} := (Z_1 - \psi_1, Z_2 - \psi_2, \ldots, Z_n - \psi_n) \in \mathbf{IR}^M$. If $\inf(W_i) \ge 0$ and $\inf(Z_i - \psi_j) \ge 0$ for $1 \le i \le M$, $1 \le j \le M$, the quadratic programming problem (2.1) has an optimal solution $z \in \mathbf{R}^M$.*

We now consider the fully automatic computer generation of a sequence of sets $\{U^{(i)}\}$, $i = 0, 1, \ldots$, which consists of subsets of $H_0^1(\Omega)$, in Section 2.

We present an iterative procedure for generating $\{U^{(i)}\}_{i=0,1,\ldots}$. For $i = 0$, we choose appropriate initial values $w_h^{(0)} \in K_h$ and $\alpha_0 \in \mathbf{R}^+$, and define $U^{(0)} \subset W$ by

$$U^{(0)} = w_h^{(0)} + [\alpha_0].$$

Usually, $u_h^{(0)}$ is determined as

$$a\left(w_h^{(0)}, v_h - w_h^{(0)}\right) \ge \left(f\left(w_h^{(0)}\right), v_h - w_h^{(0)}\right), \qquad \forall v_h \in K_h, \quad w_h^{(0)} \in K_h. \tag{2.6}$$

This corresponds to the Galerkin approximation for (1.7).

For $U_h^{(i)} = \sum_{j=1}^{M} A_j^{(i)} \phi_j$ and $\alpha_i \in \mathbf{R}^+$, we set $U^{(i)} = U_h^{(i)} + [\alpha_i]$, $i \ge 1$. Then, we define $U_h^{(i+1)} \subset K_h$ and $\alpha_{i+1} \in \mathbf{R}^+$ according to

$$w^* - D\left(z^* - \hat{\psi}\right) = D\hat{\psi} - (f(U), \phi_j), \qquad 1 \le j \le M. \tag{2.7}$$

$$\alpha_{i+1} = \sup_{u \in U^{(i)}} C(f(u), \psi) h. \tag{2.8}$$

Here, $U_h^{(i+1)}$ is determined as the solution set of (2.7), as described above. Thus, we define

$$U^{i+1} := \left(U_h^{(i+1)} + \sum_{j=1}^{M} [-\delta, +\delta]\, \phi_j\right) + [\alpha_{i+1} + \delta]$$

and then we go back to the iteration scheme in Section 1.

3. NUMERICAL EXAMPLE

In this section, we consider the one-dimensional case. In order to verify solutions numerically, it is necessary to determine the value $C(g, \psi)$ that appears in (1.8).

Let $\Omega = (0, 1)$ and $g \in L^2$. Now, let n be a positive integer and let $h = 1/(M+1)$. We define $x_i := ih$ for $i = 0, 1, 2, \ldots, M+1$ (that is, a uniform partition of Ω) and $e_i := (x_{i-1}, x_i)$, $i = 1, 2, \ldots, M+1$. We then approximate $H_0^1(\Omega)$ by

$$V_h = \left\{v_h \in C^0(\Omega) : v_h(0) = v_h(1) = 0,\ v_h|_{e_i} \in P_1,\ i = 1, 2, 3 \ldots, M+1\right\}$$

with, as usual, P_1 representing the space of polynomials of degree ≤ 1, thus $\dim V_h = M$, and $K_h = \{v_h \in S_h \colon v_h(x_i) \ge \psi(x_i),\ i = 0, 1, 2, \ldots, M, M+1\}$.

Let $\{\phi_j\}_{j=1\cdots M}$ be a basis of V_h, such that $\phi_j(x) \ge 0$, $\forall x \in \Omega$, which satisfies

$$\phi_j(x_i) = \begin{cases} 1, & i = j, \\ 0, & i \ne j. \end{cases}$$

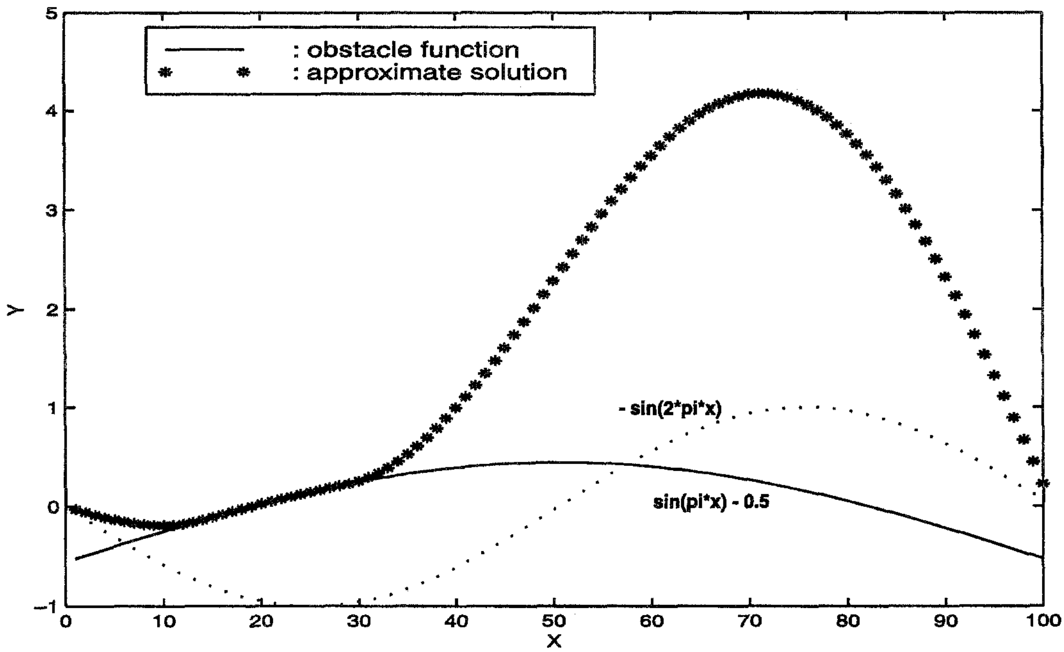

Figure 1. Approximate solution $w_h^{(0)}$.

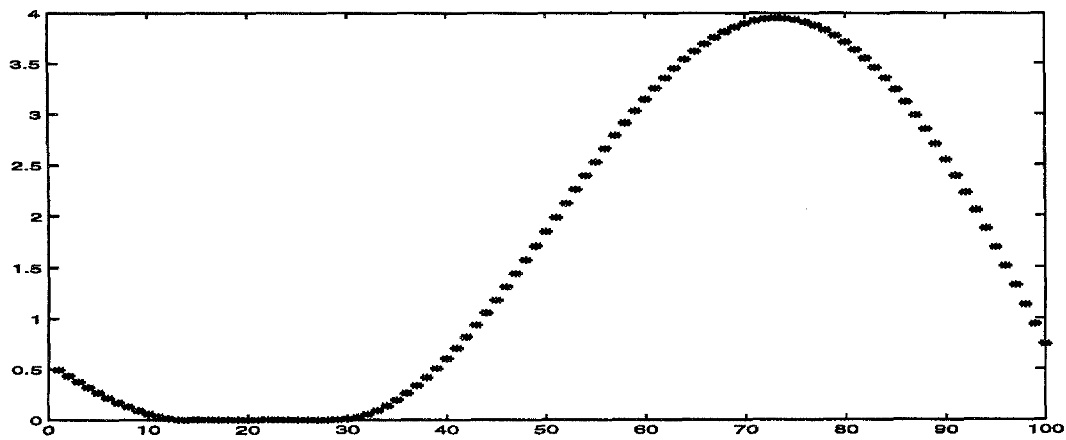

Figure 2. Approximate solution $\tilde{z} - \hat{\psi}$.

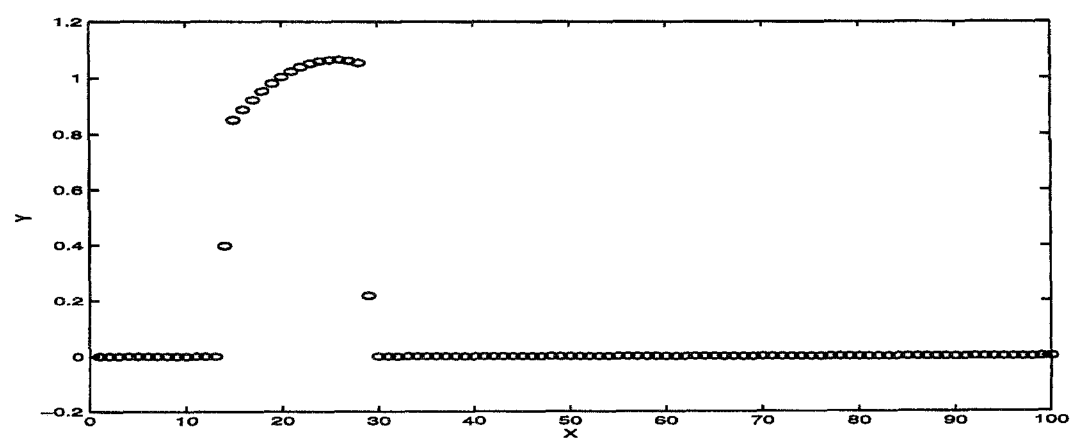

Figure 3. Approximate solution $\tilde{w}$.

THEOREM 2. *Let u and u_h be solutions of (1.1) and (1.7), respectively. If $g \in L^2(\Omega)$, then we have*

$$\|u_h - u\|_{H_0^1(\Omega)} \le C(g,\psi)h, \quad \|u_h - u\|_{L^2(\Omega)} \le C(g,\psi)\left(\frac{1}{\pi} + \frac{4\sqrt{2}}{3}\right)h^2,$$

where

$$C(g,\psi) \le \frac{1}{\pi}\sqrt{|u|^2_{H^2(\Omega)} + 2\left(\|g\|_{L^2(\Omega)} + |u|_{H^2(\Omega)}\right)\left(|u|_{H^2(\Omega)} + |\psi|_{H^2(\Omega)}\right)}.$$

Here, $|u|_{H^2(\Omega)}$ is estimated by $\|g\|_{L^2(\Omega)}$ and ψ.

We provide numerical examples of verification in the one-dimensional case following the procedure described in the previous section. We consider the case $f(u) = Qu - \sin 2\pi x$, where Q is a constant, $\psi = \sin \pi x - 0.5$.

Execution Conditions

$Q = 3$.
$\dim V_h = 100$.
Extension parameter: $\delta = 10^{-5}$.
Initial values: $w_h^{(0)}$ = Galerkin approximation (2.6). $\alpha_0 = 0$.
The form of $w_h^{(0)}$ is displayed in Figure 1.

We calculated a finite element approximate solution $(\tilde{w}, \tilde{z} - \hat{\psi})$ satisfying (2.3). Figures 2 and 3 display the shape of $\tilde{w}$ and $\tilde{z} - \hat{\psi}$.

Results

Iteration numbers: $N = 8$.
L^2-error bound := 0.019.
Maximum width of coefficient intervals in $\{A_j^{(N)}\}$ = 0.0027436946978.

REMARK. In this article, all computations based on interval arithmetic have been executed using INTLAB [4], an interval package for use MATLAB V5.3.1 [5].

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